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The small-scale structure of acceleration correlations and its role in the statistical theory of turbulent dispersion

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Some previously accepted results for the form of one- and two-particle Langrangian turbulence statistics within the inertial subrange are corrected and reinterpreted using dimensional methods and kinematic constraints. These results have a fundamental bearing on the statistical theory of turbulent dispersion.

One-particle statistics are analysed in an inertial frame $\mathscr{S}$ moving with constant velocity (which is different for different realizations) equal to the velocity of the particle at the time of labelling. It is shown that the inertial-subrange form of the Lagrangian acceleration correlation traditionally derived from dimensional arguments constrained by the property of stationarity, $\mathscr{C}_0^{(a)}\bar{\epsilon}/\tau$, where $\mathscr{C}_0^{(a)}$ is a universal constant, $\bar{\epsilon}$ is the mean rate of dissipation of turbulence kinetic energy and τ is the time lag, is kinematically inconsistent with the corresponding velocity statistics unless $\mathscr{C}_0^{(a)} = 0$. On the other hand, velocity and displacement correlations in the inertial subrange are non-trivial and the traditional results are confirmed by the present analysis. Remarkably, the universal constant $\mathscr{C}_0$ which characterizes these latter statistics in the inertial subrange is shown to be entirely prescribed by the inner (dissipation scale) acceleration covariance; i.e. there is no contribution to velocity and displacement statistics from inertial-subrange acceleration structure, but rather there is an accumulation of small-scale effects.

In the two-particle case the (cross) acceleration covariance is deduced from dimensional arguments to be of the form $\bar{\epsilon}t_1^{-1}\mathscr{R}_2(t_1/t_2)$ in the inertial subrange. In contrast to the one-particle case this is non-trivial since the two-particle acceleration covariance is non-stationary and there is therefore no condition which constraints $\mathscr{R}_2$ to a form which is kinematically inconsistent with the corresponding velocity and displacement statistics. Consequently it is possible for two-particle inertial-subrange acceleration structure to make a non-negligible contribution to relative velocity and dispersion statistics. This is manifested through corrections to the universal constant appearing in these statistics, but does not otherwise affect inertial-subrange structure. Nevertheless, these corrections destroy the simple correspondence between relative- and one-particle statistics traditionally derived by *assuming* that two-particle acceleration correlations are negligible within the inertial subrange.

A simple analytic expression which is proposed as an example of the form of $\mathscr{R}_2$ provides an excellent representation in the inertial subrange of Lagrangian stochastic simulations of relative velocity and displacement statistics.

1. Introduction

The statistical basis for turbulent dispersion goes back to Taylor's (1921) development of a kinematic relation between the dispersion of independent marked fluid particles and the Lagrangian velocity correlation function. Batchelor (1949) extended that result to three dimensions and developed the connection between single-particle dispersion in a fixed reference frame and the mean concentration field. Later, Batchelor (1950, 1952) identified the importance of two-particle statistics in describing the process of relative dispersion (i.e. the separation of particle pairs or the dispersion of a cloud of material relative to its centre-of-mass) and developed kinematic relations between particle-pair velocity statistics and the mean-square concentration field. He also showed how the set of hypotheses about the small-scale structure of turbulent motion which were introduced by Kolmogorov (1941) can be applied to relative dispersion in the so-called inertial subrange (see §2 for a brief outline of Kolmogorov's theory).

These similarity hypotheses also apply to the statistics of an ensemble of independent particles each of which is considered in an inertial frame $\mathscr{S}$ which moves with constant velocity equal to the velocity of that particle at the time of labelling, say $t = 0$. There are great similarities between this conditional (on the initial velocity) one-particle motion and two-particle relative motion. However, in order to address the two, it is necessary to consider the nature of acceleration statistics. Novikov (1963) (see also Monin & Yaglom 1975, p. 546; Gifford 1982, 1983; Sawford 1984) *assumed* that the two-particle acceleration covariance can be neglected (compared to the one-particle covariance) and so derived an equivalence between two-particle relative velocity and dispersion statistics and the corresponding conditional one-particle statistics. On the other hand, Thomson (1990) inferred from exact small-time expansions (see §4.1) and numerical calculations with a Lagrangian stochastic model of two-particle motion that the two-particle acceleration covariance cannot be ignored and argued that this finding holds generally.

Here we use exact kinematic relationships to explore the connection between the similarity forms for acceleration, velocity and displacement statistics. Although we are primarily interested in two-particle or relative statistics, we first present an analysis of one-particle acceleration statistics, showing how the direct application of Kolmogorov's theory through dimensional analysis produces inertial-subrange results inconsistent with the kinematic relations between acceleration and velocity statistics, with the result that the leading-order term for the acceleration covariance in the inertial subrange must vanish. In fact the velocity inertial-subrange structure function is determined by the dissipation range of the acceleration. Most of the one-particle results are not new, but do not appear to be well known and in any case provide a useful parallel and contrast with the *two*-particle case. They additionally allow a rational consideration of the infinite-Reynolds-number limit and its connection with current stochastic models of dispersion.

Because the relative acceleration of two particles is not stationary, the two-particle two-time acceleration covariance is not just a function of the lag but of both time variables and is therefore not fully constrained in the inertial range by the requirement of kinematic consistency with velocity statistics. There is therefore no reason to ignore the contribution of the two-particle acceleration covariance to the inertial-subrange velocity and displacement statistics. Indeed, we are able to make some deductions about the form of the two-particle acceleration covariance and to show that its contribution to these statistics in the inertial subrange is of the same order as that of the one-particle acceleration covariance. We thus demonstrate that relative dispersion is more complicated than hitherto assumed and in particular that there is no simple connection between relative velocity and dispersion statistics and the corresponding conditioned one-particle statistics as proposed by Novikov (1963).

2. Background and definitions

2.1. *Kolmogorov similarity theory*

Only the idealized problem dealing with homogeneous, isotropic and stationary turbulence will be considered. However, because of the rather general analysis which we undertake, significant emphasis falls on the small-scale and small-time behaviour. Thus, for the essentially local quantities in which we are interested these restrictions are not too severe, and are essentially encapsulated in Kolmogorov's (1941) concept of local isotropy.

We follow the ideas developed by Kolmogorov (1941) in assuming that there is an equilibrium range of (small-scale) motion governed only by the viscosity of the fluid, ν, and the mean rate of dissipation of turbulence kinetic energy, $\bar{\epsilon}$. The velocity fluctuations at any point can be characterized by the Eulerian root-mean-square velocity, σ, and a lengthscale, L, which thus represent the energy-containing scales (or large scales) of motion. Empirically, it is found (Batchelor 1953, p. 103) that although dissipation is effected by viscosity at the smallest scales of motion, the dissipation rate is determined by the largest scales of motion. In particular, it is found that $\bar{\epsilon} \approx \sigma^3/L$; here we choose to define the lengthscale as

$$L = \sigma^3/\bar{\epsilon} \tag{2.1}$$

and a corresponding timescale as

$$t_L = L/\sigma. \tag{2.2}$$

Accelerations associated with the energetic eddies are of order

$$a_L = \bar{\epsilon}/\sigma = (\bar{\epsilon}/t_L)^{\frac{1}{2}}.$$

A second set of scales can be defined in terms of the governing parameters of the equilibrium range, $\bar{\epsilon}$ and ν. These are

$$\left.\begin{aligned} \eta &= (\nu^3/\bar{\epsilon})^{\frac{1}{4}}, \\ t_\eta &= (\nu/\bar{\epsilon})^{\frac{1}{2}}, \\ v_\eta &= (\nu\bar{\epsilon})^{\frac{1}{4}} \quad (= (\bar{\epsilon}t_\eta)^{\frac{1}{2}}), \\ a_\eta &= (\bar{\epsilon}^3/\nu)^{\frac{1}{4}} \quad (= (\bar{\epsilon}/t_\eta)^{\frac{1}{2}}), \end{aligned}\right\} \tag{2.3}$$

and are known as the Kolmogorov length, time, velocity and acceleration microscales respectively. They characterize those scales of motion at which viscosity converts turbulence kinetic energy into heat. This part of the equilibrium range is known as the dissipation subrange.

Finally, there is a subrange of motions within the equilibrium range, but with scales so much larger than the dissipation scales that viscosity is not important there, which is characterized solely by $\bar{\epsilon}$. This region is known as the inertial subrange, and is the focus of the present investigation. It involves times and lengths (or separations Δ) such that $t_\eta \ll t \ll t_L$ and $\eta \ll \Delta \ll L$. Refinements to Kolmogorov's ideas (Novikov & Stewart 1964; Frisch, Sulem & Nelkin 1978) are not warranted for the level of discussion here.

The turbulence Reynolds number, Re, based on the large scales is

$$Re = \sigma L/\nu \tag{2.4}$$

and here is taken to be very large, $Re \gg 1$. In fact, we have in mind in this work the limit $\nu \rightarrow 0$. The micro- and large scales are related through the Reynolds number by

$$t_\eta = Re^{-\frac{1}{2}} t_L, \quad \eta = Re^{-\frac{3}{4}} L, \quad v_\eta = Re^{-\frac{1}{4}} \sigma, \quad a_\eta = Re^{\frac{1}{4}} a_L \tag{2.5}$$

2.2. *Lagrangian statistics and kinematic constraints*

To treat relative motions, particle-pair trajectories in each of an ensemble of turbulent flows must be considered. This means describing at time t the joint positions, $\boldsymbol{x}^{(1)}(\boldsymbol{x}_0^{(1)}, t)$ and $\boldsymbol{x}^{(2)}(\boldsymbol{x}_0^{(2)}, t)$ and velocities, $\boldsymbol{u}^{(1)}(\boldsymbol{x}_0^{(1)}, t)$ and $\boldsymbol{u}^{(2)}(\boldsymbol{x}_0^{(2)}, t)$, for each particle (labelled one and two in parentheses) when given the initial positions, $\boldsymbol{x}_0^{(1)}$ and $\boldsymbol{x}_0^{(2)}$ respectively, at $t = 0$. The initial velocities are assumed to be an unbiased random sample of the turbulence. Therefore $\boldsymbol{u}_0^{(1)}$ and $\boldsymbol{u}_0^{(2)}$ are random variables which differ from realization to realization but are described by well-defined *Eulerian* statistics. The statistics describing the pair trajectories as functions of time and initial position are *Lagrangian* statistics.

We focus on the following difference variables because these highlight dependence on local or small-scale structure of the turbulence. For velocity we define

$$\mathscr{u}^{(1)} = \boldsymbol{u}^{(1)} - \boldsymbol{u}_0^{(1)}, \tag{2.6}$$

while for displacements

$$\mathscr{x}^{(1)} = \boldsymbol{x}^{(1)} - \boldsymbol{u}_0^{(1)} t - \boldsymbol{x}_0^{(1)}, \tag{2.7}$$

and likewise for particle two. The statistics of $\mathscr{x}^{(1)}$ and $\mathscr{u}^{(1)}$ reflect the intrinsic nature of the small-scale turbulence within each flow and not the (large-scale) variability of velocity at $\boldsymbol{x}_0^{(1)}$ between different flows in the *ensemble*. Batchelor (1952) indicates how to analyse the dispersion in terms of the difference variables. Similarly, the relative velocity and displacement variables, $\mathscr{u}^{(\mathrm{r})}$ and $\mathscr{x}^{(\mathrm{r})}$, are defined as

$$\mathscr{u}^{(\mathrm{r})} = \mathscr{u}^{(2)} - \mathscr{u}^{(1)} \tag{2.8}$$

and

$$\mathscr{x}^{(\mathrm{r})} = \mathscr{x}^{(2)} - \mathscr{x}^{(1)}, \tag{2.9}$$

with all subsequent pairwise relative variables denoted by superscript r in parenthesis. The physical separation of particles is $\boldsymbol{\Delta} = \boldsymbol{x}^{(2)} - \boldsymbol{x}^{(1)}$.

Our aim is to describe the statistics of the pairwise-relative variables as they develop in time. By definition, the differences all vanish initially but after sufficient time the root-mean-square velocity fluctuations become constant. Therefore, the difference statistics are clearly not stationary. Moreover, spatial symmetry requires that

$$\langle \mathscr{u}^{(1)} \rangle = \langle \mathscr{u}^{(\mathrm{r})} \rangle = \boldsymbol{0} \quad \text{and} \quad \langle \mathscr{x}^{(1)} \rangle = \langle \mathscr{x}^{(\mathrm{r})} \rangle = \boldsymbol{0}, \tag{2.10}$$

where $\langle \ldots \rangle$ denotes an ensemble average (at time t); thus the first information of any consequence is contained in the covariances (the second moments). No higher-order statistics than these will be considered. For instance the two-time pairwise relative-velocity covariance is written as

$$D_{ij}^{(\mathrm{r})}(t_1, t_2) = \langle \mathscr{u}_i^{(\mathrm{r})}(t_1)\, \mathscr{u}_j^{(\mathrm{r})}(t_2) \rangle \tag{2.11}$$

for some tensor $\boldsymbol{D}^{(\mathrm{r})}$.

Similarly, the pairwise relative-acceleration covariance and the relative-displacement covariance are

$$R_{ij}^{(\mathrm{r})}(t_1, t_2) = \langle a_i^{(\mathrm{r})}(t_1)\, a_j^{(\mathrm{r})}(t_2) \rangle \tag{2.12}$$

and
$$F_{ij}^{(\mathrm{r})}(t_1, t_2) = \langle x_i^{(\mathrm{r})}(t_1)\, x_j^{(\mathrm{r})}(t_2)\rangle. \tag{2.13}$$

Corresponding respective, one- and two-particle moments, $D_{ij}(t_1, t_2)$ and $D_{2ij}(t_1, t_2)$ etc., where subscript two denotes the two-particle form, are defined in an analogous way. For example

$$D_{ij}(t_1, t_2) = \langle u_i^{(1)}(t_1)\, u_j^{(1)}(t_2)\rangle \quad \text{and} \quad D_{2ij}(t_1, t_2) = \langle u_i^{(1)}(t_1)\, u_j^{(2)}(t_2)\rangle.$$

The one-particle statistics so defined, involve motion in an inertial frame $\mathcal{S}$ moving with the initial velocity of the particle (Monin & Yaglom 1975, p. 533). They are also sometimes referred to as conditional one-particle statistics (Smith 1968) because they are equivalent to those statistics obtained by sampling only those particles which are initially at rest in a fixed reference frame. For simplicity, dependence on initial position has been omitted from (2.11)–(2.13). Actually, because of homogeneity, only the two-particle covariances depend on initial position, and then only on the separation of the pair,

$$\boldsymbol{\Delta}_0 = \boldsymbol{x}_0^{(2)} - \boldsymbol{x}_0^{(1)}.$$

To understand the structure of $\boldsymbol{D}^{(\mathrm{r})}$ it is helpful to consider the expanded form of the covariances. For example,

$$\langle u_i^{(\mathrm{r})} u_j^{(\mathrm{r})}\rangle = \langle u_i^{(1)} u_j^{(1)}\rangle + \langle u_i^{(2)} u_j^{(2)}\rangle - \langle u_i^{(1)} u_j^{(2)}\rangle - \langle u_i^{(2)} u_j^{(1)}\rangle. \tag{2.14}$$

The first two terms are one-particle statistics and the second two, two-particle statistics. Because of homogeneity, the one-particle statistics are independent of the initial positions and therefore the first two terms in (2.14) are equivalent. The two-particle statistics depend, however, upon $\boldsymbol{\Delta}_0$ (if $|\boldsymbol{\Delta}_0|$ were arbitrarily large little correlation between the particles would be expected) and by symmetry both two-particle terms in (2.14) are equivalent. Thus

$$D_{ij}^{(\mathrm{r})}(t_1, t_2) = 2D_{ij}(t_1, t_2) - 2D_{2ij}(t_1, t_2). \tag{2.15}$$

Relative acceleration and displacement statistics can similarly be written in terms of one- and two-particle contributions.

Consideration of the acceleration covariance is central to our task because, from the definition of the acceleration,

$$R_{ij}^{(\mathrm{r})}(t_1, t_2) = \frac{\partial^2}{\partial t_1\, \partial t_2} D_{ij}^{(\mathrm{r})}(t_1, t_2).$$

Then, since by definition (2.11)

$$D_{ij}^{(\mathrm{r})}(t_1, 0) = D_{ij}^{(\mathrm{r})}(0, t_2) = 0,$$

integration gives

$$D_{ij}^{(\mathrm{r})}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} R_{ij}^{(\mathrm{r})}(\tau_1, \tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2. \tag{2.16}$$

There are equivalent expressions for the uncoupled one- and two-particle statistics. Moreover, analogous calculation of the displacement statistics is possible from the velocity covariance and, therefore, ultimately from the acceleration covariance. In particular,

$$F_{ij}^{(\mathrm{r})}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} D_{ij}^{(\mathrm{r})}(\tau_1, \tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2. \tag{2.17}$$

The significance of *two-time* covariances is clear when the process is examined in this way because the integrals, even when $t_1 = t_2$, are calculated over the entire two-time domain $(\tau_1, \tau_2) \in [0, t_1] \times [0, t_2]$. (As we shall show, the importance of two-time statistics is obscured when considering one-particle accelerations because of stationarity; then only time *differences* (lags) are important.)

3. One-particle statistics

3.1. *Kinematic constraints*

The conditions of stationarity and isotropy (a one-particle isotropic tensor being simply a scalar function multiplying the identity tensor; thus $D_{ij} = D\delta_{ij}$ etc.) enable the one-particle equation corresponding to (2.16) to be expressed as

$$D(t_1, t_2) = \int_0^{t_1} (t_1 - \tau) R(\tau)\, d\tau + \int_0^{t_2} (t_2 - \tau) R(\tau)\, d\tau - \int_0^{t_2 - t_1} (t_2 - t_1 - \tau) R(\tau)\, d\tau, \quad (3.1)$$

and for the mean-square velocity increment with zero lag,

$$D(t, t) = 2 \int_0^{t} (t - \tau) R(\tau)\, d\tau. \quad (3.2)$$

The corresponding differential form of (3.2) is

$$R(\tau) = \frac{1}{2} \frac{d^2}{d\tau^2} D(\tau, \tau). \quad (3.3)$$

It is the latter velocity-increment structure function for zero lag that is most often used (Monin & Yaglom 1975) to describe one-particle statistics and we shall proceed to write $D(\tau, \tau)$ as $D(\tau)$.

Equation (3.2) may be used to examine the kinematic consequences of various representations for R. The aim is to have a self-consistent asymptotic structure for the velocity covariance based on some asymptotic form for R.

For large times such that $t/t_L \to \infty$, (3.2) can be written

$$2\sigma^2 = 2t \int_0^{t} R(\tau)\, d\tau - 2 \int_0^{t} \tau R(\tau)\, d\tau, \quad (3.4)$$

where the limiting value for $D(t, t)$ follows from the fact that the velocities at time t are uncorrelated with initial velocities in that limit. Clearly the acceleration covariance must approach zero fast than τ^{-2} as $\tau \to \infty$ in order that the integrals be finite. In fact, it is anticipated that the covariance vanishes exponentially fast in this limit; thus the implications of (3.4) are that

$$\int_0^{\infty} R(\tau)\, d\tau = 0 \quad \text{and} \quad \int_0^{\infty} \tau R(\tau)\, d\tau = -\sigma^2 \quad (3.5)$$

are two kinematic constraints which must independently be consistent with the expansions that are used to represent R.

Figure 1 is a sketch of the anticipated behaviour of R for a moderate range of lags, including the dissipation range. The acceleration covariance is expected to be positive for small enough lags, but for some range of lags must necessarily be negative because of (3.5). Furthermore, the area between the curve and the (lag) τ-axis for that portion where the curve is above the axis is the same as the area enclosed by the curve and the axis when it falls below the axis.

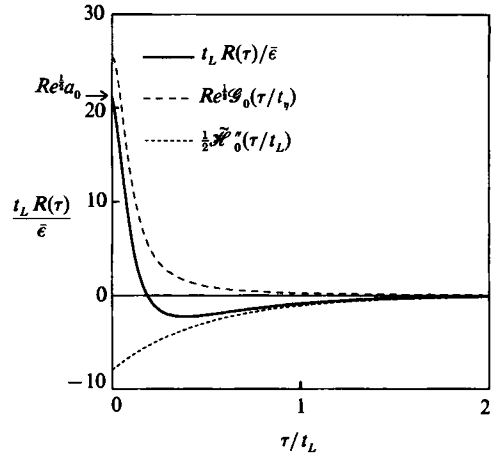

FIGURE 1. Schematic of the acceleration covariance as a function of the lag. The first terms from an inner and outer representation, $\mathscr{G}_0$ and $\tilde{\mathscr{H}}_0''$ respectively, are also shown (see §3.3). *Re* is taken to be 400, $\mathscr{C}_0$ to be $2\surd 2$ and a_0 is an $O(1)$ numerical factor, here taken to be 1.1.

3.2. *Inertial-range behaviour*

Following Kolmogorov we consider the behaviour of the various statistical quantities for the time-lag regime where $t_\eta \ll \tau \ll t_L$ and suppose that these quantities then only depend upon $\bar{\epsilon}$ and τ. The particular physical relevance is that these are generic properties, independent of the details of both small and large scales and describe a universal transition between the two extremes. This regime is generally (Monin & Yaglom 1975) denoted the 'inertial subrange' but for the remainder of this paper shall simply be known as the 'inertial range'.

The inertial-range results for acceleration, velocity increment and displacement covariances are (from dimensional analysis)

$$\left.\begin{aligned} R(\tau) &\approx \mathscr{C}_0^{(a)}\,\bar{\epsilon}\tau^{-1}+o(\tau^{-1}),\\ D(\tau) &\approx \mathscr{C}_0^{(u)}\,\bar{\epsilon}\tau+o(\tau),\\ \text{and}\qquad F(\tau) &\approx \mathscr{C}_0^{(x)}\,\bar{\epsilon}\tau^3+o(\tau^3)\end{aligned}\right\} \tag{3.6}$$

for $t_\eta \ll \tau \ll t_L$, where $\mathscr{C}_0^{(a)}$, $\mathscr{C}_0^{(u)}$ and $\mathscr{C}_0^{(x)}$ are dimensionless constants and are supposed $O(1)$. In particular $\mathscr{C}_0^{(u)}$ is more usually (Monin & Yaglom 1975, p. 359) written as $\mathscr{C}_0$. Its value is very uncertain with estimates by Hanna (1981) and Anand & Pope (1985) in the range $\mathscr{C}_0 = 4\pm 2$.

It turns out that there is a very simple relationship between $\mathscr{C}_0^{(u)}$ and $\mathscr{C}_0^{(x)}$, namely $\mathscr{C}_0^{(u)} = 3\mathscr{C}_0^{(x)}$; however, there is an immediate difficulty in relating $\mathscr{C}_0^{(u)}$ $(=\mathscr{C}_0)$ to $\mathscr{C}_0^{(a)}$ as is evident from (3.2), which cannot be satisfied by the inertial-range forms in (3.6). If $\mathscr{C}_0^{(a)} \neq 0$ then the inconsistency suggests modifying the inertial-range form for velocity to either of

$$D(\tau) \approx \mathscr{C}_0^{(u)}\,\bar{\epsilon}\tau\log\left(\frac{\tau}{t_\eta}\right) \quad \text{or} \quad \mathscr{C}_0^{(u)}\,\bar{\epsilon}\tau\log\left(\frac{\tau}{t_L}\right) \tag{3.7}$$

or perhaps a linear combination of these. Note that any such form differs from another only by a linear term in τ whose coefficient depends on *Re* and, furthermore, that (3.3) requires $\mathscr{C}_0^{(u)} = 2\mathscr{C}_0^{(a)} = \mathscr{C}_0$. Moreover, the linear term does not affect the acceleration inertial-range form because it vanishes upon differentiation in (3.3). It is essentially the $\tau\log\tau$ term which is consistent with the inverse-τ acceleration covariance, but the extra parts of (3.7) are required for dimensional consistency.

However, the principles defining the inertial range are violated by (3.7) where it is apparent the simultaneous independence of the parameters t_η and t_L is not possible. Moreover, the prediction that $\mathscr{C}_0^{(a)} = \frac{1}{2}\mathscr{C}_0$ (which is necessarily positive) is inconsistent with the behaviour illustrated in figure 1. It is very unlikely that the curve intersects the axis twice as would be required by positive $\mathscr{C}_0^{(a)}$. Therefore, it must be that $\mathscr{C}_0^{(a)} = 0$ which means that the inertial-range accelerations are trivial, asymptotically smaller than the dimensional prediction $O(\bar{\epsilon}\tau^{-1})$. This fact has been observed by Kraichnan (1966*a*) but it is not generally well known. For example Monin & Yaglom (1975), equation (21.54) on page 370 propose otherwise.

3.3 *Asymptotic expansions*

The foregoing work illustrates the difficulty with simple dimensional arguments; it is evident that a more elaborate structure, perhaps as afforded by asymptotic expansions, is required to provide a self-consistent picture. The goal is to understand why the 'inertial-range' constant for acceleration covariance vanishes and what form the acceleration covariance takes. The approach adopted here has some parallels with the rational asymptotic analysis of a turbulent boundary layer, as given by Mellor (1972) in that inertial-range properties are considered as matching properties of small- and large-scale representations. However, while Mellor bases his discussion on Eulerian quantities and uses the Navier–Stokes equations we consider Lagrangian statistics and only attempt to use the kinematic relations, equations (3.5). In our case there is the additional novelty that we do *not* expect an inertial range for the acceleration covariance, which prompts new questions regarding the matching interpretation of the inertial range.

An expansion which describes the behaviour of the dissipation range, i.e. the small scales, but not the energy-containing scales (large scales) shall be the inner expansion. The acceleration covariance as illustrated in figure 1 can generally be represented in the inner region by $R = \bar{\epsilon}t_\eta^{-1}\mathscr{R}$, where

$$\mathscr{R}\left(\frac{\tau}{t_\eta};Re\right) = \mathscr{G}_0\left(\frac{\tau}{t_\eta}\right) + \mathscr{G}_1\left(\frac{t}{t_\eta}\right)\delta_1(Re) + \mathscr{G}_2\left(\frac{\tau}{t_\eta}\right)\delta_2(Re) + \dots \tag{3.8}$$

and the functions $\mathscr{G}_i(\xi)$ and $\delta_i(Re)$ are, in some sense, to be determined. Notice that the leading-order term is independent of t_L (and therefore Re) and the subsequent terms represent Re-dependent corrections. Furthermore, $1 \gg \delta_1 \gg \delta_2 \gg \dots$ and, formally, τ/t_η is of $O(1)$.

Similarly, the large-scale properties are described by $R = \bar{\epsilon}t_L^{-1}\tilde{\mathscr{R}}$ and an outer expansion:

$$\tilde{\mathscr{R}}\left(\frac{\tau}{t_L};Re\right) = \tilde{\mathscr{G}}_0\left(\frac{\tau}{t_L}\right) + \tilde{\mathscr{G}}_1\left(\frac{\tau}{t_L}\right)\tilde{\delta}_1(Re) + \tilde{\mathscr{G}}_2\left(\frac{\tau}{t_L}\right)\tilde{\delta}_2(Re) + \dots, \tag{3.9}$$

where $1 \gg \tilde{\delta}_1 \gg \tilde{\delta}_2 \gg \dots$. In like manner all quantities have analogous inner and outer expansions. In particular we note that the velocity-increment-covariance outer expansion corresponds to $D = \bar{\epsilon}t_L\tilde{\mathscr{D}}$ with

$$\tilde{\mathscr{D}}\left(\frac{\tau}{t_L};Re\right) = \tilde{\mathscr{H}}_0\left(\frac{\tau}{t_L}\right) + \tilde{\mathscr{H}}_1\left(\frac{\tau}{t_L}\right)\tilde{\delta}_1(Re) + \tilde{\mathscr{H}}_2\left(\frac{\tau}{t_L}\right)\tilde{\delta}_2(Re) + \dots \tag{3.10}$$

and consequently from (3.3)

$$\tilde{\mathscr{R}}\left(\frac{\tau}{t_L};Re\right) = \tfrac{1}{2}\tilde{\mathscr{H}}_0''\left(\frac{\tau}{t_L}\right) + \tfrac{1}{2}\tilde{\mathscr{H}}_1''\left(\frac{\tau}{t_L}\right)\tilde{\delta}_1(Re) + \tfrac{1}{2}\tilde{\mathscr{H}}_2''\left(\frac{\tau}{t_L}\right)\tilde{\delta}_2(Re) + \dots. \tag{3.11}$$

We can only postulate what these asymptotic forms may be. For example, for the large-τ expansion of the inner acceleration-covariance expansion it is consistent with inertial-range hypotheses that

$$\mathscr{G}_0\left(\frac{\tau}{t_\eta}\right) = \mathscr{C}_0^{(a)}\left(\frac{\tau}{t_\eta}\right)^{-1} + \dots, \tag{3.12}$$

where $\tau/t_\eta \gg 1$. Similarly, for small τ/t_L

$$\tilde{\mathscr{H}}''_0\left(\frac{\tau}{t_\eta}\right) = 2\mathscr{C}_0^{(a)}\left(\frac{\tau}{t_L}\right)^{-1} + \dots \tag{3.13}$$

also leads to the inertial-range form for the acceleration covariance, but in this case as a small-τ expansion of the outer expansion. Thus, in the sense of Van Dyke (1975) the expansions match; or in Mellor's sense the mutual overlap of the expansions constitutes the inertial range. However, it was indicated above that an inertial range for the acceleration covariance led to an inconsistency overall (cf. (3.7)). This inconsistency can in fact be shown to correspond to a violation of the kinematic constraint (3.5): we write a uniform approximation (*for all* τ) to R as a composite expansion (Van Dyke 1975)

$$R(\tau) \approx \bar{\epsilon} t_\eta^{-1}\mathscr{G}_0\left(\frac{\tau}{t_\eta}\right) + \tfrac{1}{2}\bar{\epsilon} t_L^{-1}\tilde{\mathscr{H}}''_0\left(\frac{\tau}{t_L}\right) - \mathscr{C}_0^{(a)}\,\bar{\epsilon}\tau^{-1}. \tag{3.14}$$

Equation (3.14) is valid provided the first term decreases like $\mathscr{C}_0^{(a)}\,\bar{\epsilon}\tau^{-1}$ as $\tau/t_\eta \to \infty$ while the second term grows like $\mathscr{C}_0^{(a)}\,\bar{\epsilon}\tau^{-1}$ as $\tau/t_L \to 0$; therefore in the inertial range (matching region) the collective behaviour of the approximation is just $R \approx \mathscr{C}_0^{(a)}\,\bar{\epsilon}\tau^{-1}$. Now consider the ramifications of (3.14) in the exact integral constraints (3.5). Grouping the last two terms together, it is possible to integrate over the time-lag domain $[0, \tau)$ giving

$$\int_0^\tau \left(\tfrac{1}{2}\bar{\epsilon} t_L^{-1}\tilde{\mathscr{H}}''_0\left(\frac{\tau}{t_L}\right) - \mathscr{C}_0^{(a)}\,\bar{\epsilon}\tau^{-1}\right)\mathrm{d}\tau = \bar{\epsilon}\int_0^{\tau/t_L}\left(\tfrac{1}{2}\tilde{\mathscr{H}}''_0(\xi) - \mathscr{C}_0^{(a)}\xi^{-1}\right)\mathrm{d}\xi$$

$$\approx -\mathscr{C}_0^{(a)}\bar{\epsilon}\log\left(\frac{\tau}{t_L}\right) + \bar{\epsilon}\mathscr{g}_0 + \dots$$

for large τ where $\mathscr{g}_0$ is some $O(1)$ constant. However, integrating the remaining term from (3.14) over the same domain provides a contradiction because

$$\int_0^\tau \bar{\epsilon} t_\eta^{-1}\mathscr{G}_0\left(\frac{\tau}{t_\eta}\right)\mathrm{d}\tau = \bar{\epsilon}\int_0^{\tau/t_\eta}\mathscr{G}_0(\xi)\,\mathrm{d}\xi \approx \mathscr{C}_0^{(a)}\,\bar{\epsilon}\log\left(\frac{\tau}{t_\eta}\right) + \bar{\epsilon}\tilde{\mathscr{g}}_0 + \dots$$

when τ/t_η is large (for some $\tilde{\mathscr{g}}_0$), and when combined with the other terms gives the approximate result

$$\int_0^\infty R(\tau)\,\mathrm{d}\tau \approx \mathscr{C}_0^{(a)}\,\bar{\epsilon}\log\left(\frac{t_L}{t_\eta}\right) + \bar{\epsilon}(\tilde{\mathscr{g}}_0 + \mathscr{g}_0) \gg \bar{\epsilon}$$

after the limit $\tau \to \infty$ has been taken. Thus (3.5) can only be satisfied when $\mathscr{C}_0^{(a)} = 0$ in which case (3.14) is not a uniform approximation to R, failing to represent the inertial range, although it is satisfactory elsewhere. In fact there is no inertial range in the sense of Mellor (1972). The variation of the acceleration covariance for $t_\eta \ll \tau \ll t_L$ is actually much more complex and is always a function of both Re and τ such that it remains $\ll \bar{\epsilon}\tau^{-1}$. The higher-order terms indicated in both (3.8) and (3.9) play a role in the matching region (because the $\mathscr{G}_0$ term cannot match with the leading order of the outer expansion – thus $\mathscr{G}_1$ or subsequent terms must!). Thus the composite expansions are necessarily more complicated than (3.14) and will not be pursued any further. No useful formula for the inertial-range variation can be deduced from the expansions. This is because both expansions are non-uniform and, in the worst case possible, infinitely many terms of the same order of magnitude constitute the leading-order behaviour.

3.4. *Inertial-range behaviour of velocity and displacement*

Given that $\mathscr{C}_0^{(a)}$ vanishes it follows that $\mathscr{G}_0$ is integrable over the entire τ-domain. Thus if we substitute (3.8) into (3.2) then at leading-order in the inner range

$$D(\tau) \approx 2\bar{\epsilon}\int_0^{\tau/t_\eta} (\tau - t_\eta\,\xi)\,\mathscr{G}_0(\xi)\,\mathrm{d}\xi. \tag{3.15}$$

Now suppose $\tau \gg t_\eta$; since the integral of $\mathscr{G}_0$ exists the approximation in the *inertial* range is written as

$$D(\tau) \approx 2\bar{\epsilon}\tau \int_0^{\infty} \mathscr{G}_0(\xi)\,\mathrm{d}\xi. \tag{3.16}$$

Either matching with (3.12) or comparison with (3.6) gives the result that

$$\mathscr{C}_0^{(u)} = \mathscr{C}_0 = 2\int_0^{\infty} \mathscr{G}_0(\xi)\,\mathrm{d}\xi. \tag{3.17}$$

Thus a peculiar structure emerges: while there is only a trivial acceleration inertial range ($\equiv 0$) with no parameters describing it, the inertial range for velocity fluctuations is non-trivial and is categorized by a non-zero (presumably) universal parameter $\mathscr{C}_0^{(u)}$. Interestingly, that parameter is entirely prescribed by the inner acceleration covariance as represented by the function $\mathscr{G}_0$. What is remarkable is that $\mathscr{G}_0$ effectively only describes the dissipation range and is negligible for $\tau \gg t_\eta$, yet it follows that the dissipation-range covariance collectively determines the velocity inertial range through (3.17). The definition of $\mathscr{G}_0$ ensures, however, that $\mathscr{C}_0$ as given by (3.17) is universal and is independent of both ν and t_L. This conclusion is similar in principle to that of Lin & Reid (1963); however, their result associates $\mathscr{C}_0$ less clearly with the dissipation-range structure of the acceleration covariance.

The situation for the velocity covariance is markedly different. It is apparently consistent to have the naive leading-order linear term within the inertial range:

$$D(\tau) = \mathscr{C}_0\,\bar{\epsilon}\tau + o(\bar{\epsilon}\tau) \quad \text{for} \quad t_\eta \ll \tau \ll t_L. \tag{3.18}$$

Moreover, in the context of inner and outer expansions the leading-order term in the inner expansion tends to $\mathscr{C}_0\,\bar{\epsilon}\tau$ as τ/t_η becomes large while the leading-order outer-expansion term also tends to $\mathscr{C}_0\,\bar{\epsilon}\tau$, but now as τ/t_L becomes small. Therefore, the asymptotic structure is *not* non-uniform in the way it was for the acceleration covariance.

Note that there is no equivalent difficulty in consistency, as encountered above, when extending the analysis to the displacement covariance. To accomplish this we need the two-time velocity increment statistics: specializing to the inertial range, these follow from (Monin & Yaglom 1975, p. 533)

$$D(t_1, t_2) = \tfrac{1}{2}\mathscr{C}_0\,\bar{\epsilon}(t_1 + t_2 - |t_2 - t_1|), \tag{3.19}$$

which in the one-particle version of (2.17) integrates to give the two-time displacement statistics. The leading-order mean-square term within the inertial range is (without further assumption)

$$F(\tau) = \tfrac{1}{3}\mathscr{C}_0 \bar{\epsilon}\tau^3 + o(\bar{\epsilon}\tau^3) \quad \text{for} \quad t_\eta \ll \tau \ll t_L. \tag{3.20}$$

Thus $\mathscr{C}_0$ is not required (by kinematic consistency at least) to vanish.

3.5. *Large-Reynolds-number limit*

It is illustrative to consider the formal limit as $\nu \rightarrow 0$, i.e. $Re \rightarrow \infty$. If τ is kept fixed then for $\tau > 0$

$$R(\tau) = \tfrac{1}{2}\bar{\epsilon} t_L^{-1} \tilde{\mathscr{H}}_0''\left(\frac{\tau}{t_L}\right),$$

which is now an exact relation. For $\tau = 0$ though, the limit is undefined because $R \approx \bar{\epsilon} t_\eta^{-1} \mathscr{G}_0(0) \rightarrow \infty$. However, because $\mathscr{G}_0(\xi)$ is *integrable* along the real axis for any finite ν, the acceleration covariance can be interpreted in terms of generalized functions (Lighthill 1958) has having a δ-function at $\tau = 0$. Thus for $\tau \geqslant 0$

$$R(\tau) = \mathscr{C}_0 \bar{\epsilon} t_L^{-1} \delta\left(\frac{\tau}{t_L}\right) + \tfrac{1}{2}\bar{\epsilon} t_L^{-1} \tilde{\mathscr{H}}_0''\left(\frac{\tau}{t_L}\right) \tag{3.21}$$

is appropriate for infinite-Reynolds-number turbulence. The δ-function corresponds to a white-noise stochastic process suggesting, as is often assumed, that in the large-Re limit the Lagrangian velocity can be modelled as a Markov process. Then $\tilde{\mathscr{H}}_0$ takes the particular form

$$\tilde{\mathscr{H}}_0\left(\frac{\tau}{t_L}\right) = 2\left(1 - \exp\left(-\tfrac{1}{2}\mathscr{C}_0 \frac{|\tau|}{t_L}\right)\right).$$

To this point only the first of the integrals in (3.5) has been used to check kinematic consistency. Analysis using the second integral constraint does not lead to any further restriction on the governing constants. There are, however, some interesting differences. For example, the exact constraint that

$$\int_0^\infty R(\tau)\,\mathrm{d}\tau = 0$$

requires finite contributions from both the inner and outer regions. It is particularly illustrative to consider the infinite-Re case. The outer term, for $\tau > 0$, gives the contribution

$$\int_0^\infty R(\xi)\,\mathrm{d}\xi = [\tfrac{1}{2}\tilde{\mathscr{H}}_0'(\xi)]_0^\infty = -\tfrac{1}{2}\tilde{\mathscr{H}}_0'(0) = -\tfrac{1}{2}\mathscr{C}_0 \neq 0$$

and the exact result is only retrieved when the additional contribution from the δ-function is considered:

$$\int_0^\infty \mathscr{C}_0 \delta(\xi)\,\mathrm{d}\xi = +\tfrac{1}{2}\mathscr{C}_0.$$

Thus contributions from both the inner and outer regions are essential. This fact is contrasted by the second exact result from (3.5), with

$$\int_0^\infty \tau R(\tau)\,\mathrm{d}\tau = -\sigma^2$$

being implicitly satisfied by the outer term alone. This follows from integration by parts and since $\tilde{\mathscr{H}}_0(\infty) = 2$ (in general). Therefore, the inner region contributes only negligible corrections to the exact result.

4. Two-particle statistics

4.1. *General considerations*

The conventional approach based on dimensional analysis (Monin & Yaglom 1975, p. 546; Novikov 1963, 1989) is to argue that because the accelerations are localized in space (i.e. the Eulerian acceleration covariance decays rapidly with separation) as well as time, the two-particle Lagrangian acceleration covariance is negligible in the inertial range, or, equivalently that the relative acceleration is stationary. This means, through (2.14), that the pairwise-relative velocity covariance is effectively twice the one-particle velocity covariance and so forth. Thus, according to this approach, the pairwise-relative statistics are simply related to the one-particle statistics and the two-particle statistics are essentially redundant. However, the neglect of the two-particle acceleration covariance is an assumption which is not rigorously justified. Indeed, deeper examination, which is the endeavour of the remainder of this paper, indicates the converse; the two-particle covariance is as important as the one-particle one for significant periods of time.

The joint dispersion of two particles in a turbulent flow is more complex because the acceleration process is not stationary. Thus, from (3.14), $R_{2ij}(t_1, t_2)$ depends non-trivially upon both t_1 and t_2 and, in particular, is not just a function of the lag, $\tau = t_2 - t_1$. (Here and throughout the remaining sections we restrict attention to the dot-products of the statistical quantities of the acceleration, velocity etc., e.g. $R_2 = R_{2ii}$, $D_2 = D_{2ii}, F_2 = F_{2ii}, \ldots$.) Figure 2 shows schematically the form of the covariance. The process is not stationary because it 'remembers' the initial particle labelling. That is to say, particles that are close initially will be more strongly correlated then than after the turbulent flow has dispersed them. In fact, after a sufficiently long time has elapsed, say $t_1 \gg t_L$, the particles will effectively be independent of one another even for zero lag ($t_1 = t_2$)! In contrast, if the process were stationary and therefore the covariance simply a function of τ, it would be identical with the initial covariance whenever τ vanishes.

Evidently a new timescale is important. This scale is a measure of the rate of loss of memory of the initial conditions and must depend upon the initial separation, Δ_0. Letting $\Delta_0^2 = \Delta_{0i}\Delta_{0i}$, we are interested in inertial-range separations $\eta \ll \Delta_0 \ll L$, and then the appropriate time-scale, t_0, is

$$t_0 = \Delta_0^{\frac{2}{3}} \bar{\epsilon}^{-\frac{1}{3}}.$$

Therefore, the domain of interest (the inertial range) involves times such that

$$t_\eta \ll t_0, t_1, t_2 \ll t_L. \tag{4.1}$$

Within the time range (4.1) we distinguish two subranges, which for want of better terminology we call, following Batchelor (1950), 'small' times, such that $t_\eta \ll t_1, t_2 \ll t_0$, and 'intermediate' times, such that $t_\eta \ll t_0 \ll t_1, t_2 \ll t_L$.

For small times, $t_1, t_2 \ll t_0$, the Taylor series expansion of the two-particle velocity structure function gives (from (2.16) modified for cross-statistics)

$$D_2(t_1, t_2) = R_2^{(\mathrm{E})}(\Delta_0)\, t_1 t_2 + O(t_1^2 t_2, t_1 t_2^2),$$

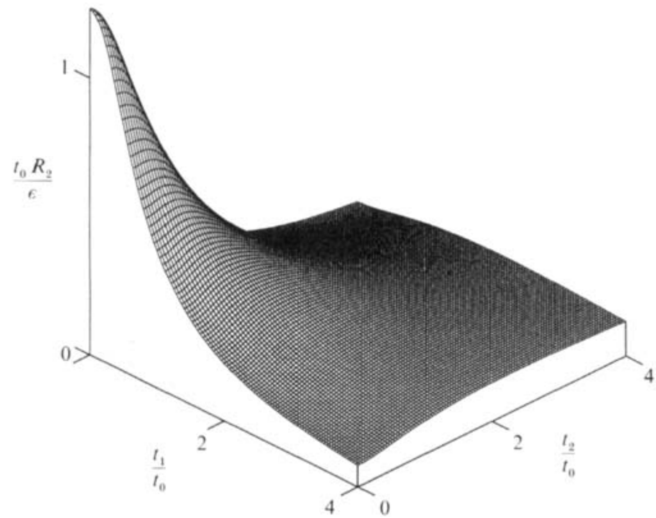

FIGURE 2. Schematic of the scalar product of the two-particle two-time acceleration covariance, $R_2 = \langle a_i^{(1)}(t_1)\, a_i^{(2)}(t_2)\rangle$, as a function of the times, t_1 and t_2. (The surface was constructed using (6.2) with $\mu = \frac{3}{2}$).

where $R_2^{(E)}(\Delta_0)$ is the two-point Eulerian acceleration correlation, i.e.

$$R_2^{(E)}(\Delta_0) = \langle a_i(\boldsymbol{x})\, a_i(\boldsymbol{x}+\Delta_0)\rangle$$

which is independent of time since the turbulence is stationary. For inertial-range separations (Monin & Yaglom 1975, p. 371)

$$R_2^{(E)}(\Delta_0) \approx \kappa\bar{\epsilon}(\Delta_0^2/\bar{\epsilon})^{-\frac{1}{3}} = \kappa\bar{\epsilon}/t_0 \quad (\eta \ll \Delta_0 \ll L)$$

where κ is a universal constant. Therefore,

$$D_2(t,t) = \kappa\bar{\epsilon}t^2/t_0 + o(t^2/t_0) \quad (\eta \ll \Delta_0 \ll L, \quad t_\eta \ll t \ll t_0) \tag{4.2}$$

Initially, therefore, D_2 is much smaller than the corresponding one-particle structure function in the inertial range (cf. (3.18)), a result which is consistent with the usual assumption that the contribution of the two-particle acceleration correlations is negligible. However, for later times such that $t \approx t_0$, $D_2(t,t) \approx \kappa\bar{\epsilon}t$, which is of the same order as the one-particle function! Thus although the expansion (4.2) is anticipated to break down at this point, it strongly suggests that the two-particle structure function increases in importance with time and is comparable with the one-particle function within the 'intermediate' part of the inertial range, $t_0 \ll t_1, t_2 \ll t_L$. As a corollary, we expect the two-particle acceleration covariance to be significant there! This argument has been advanced in less formal terms by Thomson (1990).

4.2. *Intermediate-subrange structure*

Our approach here is to deduce the forms of the two-particle statistics for the intermediate subrange by simple dimensional analysis and then to show that the kinematic constraints do not force a trivial structure on the acceleration covariances as in the one-particle case. The reason for this difference is essentially that the two-particle acceleration covariance is not subject to the additional constraint of stationarity.

We construct forms that depend only upon t_1, t_2 and $\bar{\epsilon}$. Thus we have for the acceleration covariance, the velocity structure function and displacement covariance respectively

$$\left.\begin{aligned} R_2 &= \bar{\epsilon} t_1^{-1} \mathscr{R}_2\left(\frac{t_1}{t_2}\right) + o(\bar{\epsilon} t_1^{-1}) \\ D_2 &= \bar{\epsilon} t_1 \mathscr{D}_2\left(\frac{t_1}{t_2}\right) + o(\bar{\epsilon} t_1) \quad \text{for} \quad t_0 \ll t_1, t_2 \ll t_L. \\ \text{and} \qquad F_2 &= \bar{\epsilon} t_1^3 \mathscr{F}_2\left(\frac{t_1}{t_2}\right) + o(\bar{\epsilon} t_1^3) \end{aligned}\right\} \tag{4.3}$$

These forms are further restricted by symmetry requirements ($R_2(t_1, t_2) = R_2(t_2, t_1)$ etc.) so that

$$\left.\begin{aligned} \mathscr{R}_2(\xi) &= \xi \mathscr{R}_2(\xi^{-1}) \\ \mathscr{D}_2(\xi) &= \xi^{-1} \mathscr{D}_2(\xi^{-1}) \\ \mathscr{F}_2(\xi) &= \xi^{-3} \mathscr{F}_2(\xi^{-1}) \end{aligned}\right\} \quad \text{for} \quad \xi \in (0, \infty).$$

We now examine the consistency of these forms by substituting (4.3) into the two-particle kinematic relations corresponding to (2.16). In particular,

$$D_2(t_1, t_2) = \bar{\epsilon} t_1 \mathscr{D}_2\left(\frac{t_1}{t_2}\right) = \bar{\epsilon} \int_0^{t_1} \int_0^{t_2} \tau_1^{-1} \mathscr{R}_2\left(\frac{\tau_1}{\tau_2}\right) \mathrm{d}\tau_1 \, \mathrm{d}\tau_2. \tag{4.4}$$

After some algebra, an integration by parts, and using the symmetry properties, the kinematic constraint (4.4) reduces to

$$\mathscr{D}_2(\xi) = \xi^{-1} \int_0^{\xi} \chi^{-1} \mathscr{R}_2(\chi) \, \mathrm{d}\chi + \int_0^{\xi^{-1}} \chi^{-1} \mathscr{R}_2(\chi) \, \mathrm{d}\chi. \tag{4.5}$$

It remains to be shown that the integrals in (4.5) are proper for non-trivial $\mathscr{R}_2(\chi)$ in order that $\mathscr{D}_2(\xi)$ be well-defined. Note that (4.5) clearly embodies the symmetry requirement.

The potential problem points for (4.5) are the end points of integration, i.e. as $\chi \rightarrow 0$ or ∞, since $\mathscr{R}_2$ is smooth and no internal regions of the domain can cause integration difficulties. Because of symmetry we need only consider one of these cases, $\chi \rightarrow 0$. In Appendix A we argue that $\mathscr{R}_2(\chi)$ vanishes more rapidly than χ as $\chi \rightarrow 0$

$$\lim_{\chi \rightarrow 0} \mathscr{R}_2(\chi) \, \chi^{-1-\lambda} = \text{const.} \tag{4.6a}$$

or, equivalently,

$$\lim_{\chi \rightarrow \infty} \mathscr{R}_2(\chi) \, \chi^{\lambda} = \text{const.} \tag{4.6b}$$

for some $\lambda > 0$. With restriction (4.6) on $\mathscr{R}(\chi)$ the integrals in (4.5) are proper and the inertial-range forms for acceleration and velocity structure functions are completely self-consistent.

Note that the constraint (4.6a) on the form of $\mathscr{R}_2(\chi)$ ensures that

$$D_2(t_1, 0) = D_2(0, t_2) = 0 \, \forall \, t_1, t_2$$

respectively, as in required by definition. We conclude that there is no kinematic reason for the two-point acceleration covariance to be trivial in the inertial range.

As was the case for one-particle inertial-range statistics the displacement and velocity covariances are also completely consistent in the intermediate subrange. Thus using (4.3) and integrating as in (2.17) we find

$$\mathscr{F}_2(\xi) = \frac{1}{2}\bigg(\xi^{-1}\int_0^{\xi^{-1}} \chi^{-1}\mathscr{R}_2(\chi)\,\mathrm{d}\chi + \xi^{-2}\int_0^{\xi}\chi^{-1}\mathscr{R}_2(\chi)\,\mathrm{d}\chi$$
$$-\tfrac{1}{3}\int_0^{\xi^{-1}}\mathscr{R}_2(\chi)\,\mathrm{d}\chi - \tfrac{1}{3}\xi^{-3}\int_0^{\xi}\mathscr{R}_2(\chi)\,\mathrm{d}\chi\bigg). \quad (4.7)$$

which is well defined in the same sense as (4.5).

For dispersion, the important results are generally concerned with mean-square quantities along the diagonal of the two-time plane, $t_1 = t_2$. Then we have

$$\langle a_i^{(1)}(t)\,a_i^{(2)}(t)\rangle = R_2(t,t) \approx \mathscr{R}_2(1)\,\bar{\epsilon}t^{-1} \quad (t_0 \ll t \ll t_L); \quad (4.8a)$$

$$\langle u_i^{(1)}(t)\,u_i^{(2)}(t)\rangle = D_2(t,t) \approx 2\mathscr{C}_1\,\bar{\epsilon}t \quad (t_0 \ll t \ll t_L), \quad (4.8b)$$

where, from (4.5),
$$\mathscr{C}_1 = \int_0^1 \chi^{-1}\mathscr{R}_2(\chi)\,\mathrm{d}\chi$$

is a universal constant. It also follows from (4.5) that

$$\mathscr{D}_2(0) = \int_0^\infty \chi^{-1}\mathscr{R}_2(\chi)\,\mathrm{d}\chi$$

and
$$\mathscr{D}_2'(\xi) \approx \xi^{\lambda-1} \quad \text{as} \quad \xi \to 0.$$

Finally, from (4.7) we have

$$\langle x_i^{(1)}(t)\,x_i^{(2)}(t)\rangle = F_2(t,t) \approx (\mathscr{C}_1 - \tfrac{1}{3}\mathscr{C}_2)\,\bar{\epsilon}t^3 \quad \text{for} \quad t_0 \ll t \ll t_L, \quad (4.8c)$$

where
$$\mathscr{C}_2 = \int_0^1 \mathscr{R}_2(\chi)\,\mathrm{d}\chi,$$

$$\mathscr{F}_2(\xi) \approx \tfrac{1}{2}\mathscr{D}_2(0)\,\xi^{-1} \quad \text{as} \quad \xi \to 0,$$

and
$$\mathscr{F}_2'(\xi) \approx -\tfrac{1}{2}\mathscr{D}_2(0)\,\xi^{-2} \quad \text{as} \quad \xi \to 0.$$

These results are non-trivial provided that $\mathscr{C}_1$ and $\mathscr{C}_2$ do not vanish (or cancel in (4.8*c*)), details which cannot be verified by simple dimensional analysis.

An intriguing difference from the analogous one-particle equations,

$$D(t,t) \approx 3\mathscr{C}_0\,\bar{\epsilon}t, \quad F(t,t) \approx \mathscr{C}_0\,\bar{\epsilon}t^3$$

when $t_\eta \ll t \ll t_L$, is that whereas the characteristic constant, $\mathscr{C}_0$, is determined explicitly by the small-scale dissipation-subrange properties (3.17) of the one-particle dynamics, the two-particle $\mathscr{C}_1$ is determined explicitly by the larger scale inertial-range properties (an integral involving $\mathscr{R}_2$). Despite the disparity in interpretation of the formally $O(1)$ constants $\mathscr{C}_0$ and $\mathscr{C}_1$, it is particularly important to note that both processes lead to equal order-of-magnitude estimates of dispersion for times $t \gg t_0$. Therefore it is clear that an account of relative dispersion must include both processes.

The *correlation coefficients* corresponding to (4.8*b*) and (4.8*c*) are, using the one-particle results above,

$$\frac{\langle u_i^{(1)}(t)\,u_i^{(2)}(t)\rangle}{\langle u_j^{(1)}(t)^2\rangle^{\frac{1}{2}}\langle u_k^{(2)}(t)^2\rangle^{\frac{1}{2}}} = \frac{D_2(t,t)}{D(t,t)} = \frac{2\mathscr{C}_1}{3\mathscr{C}_0} \quad (t_0 \ll t \ll t_L) \quad (4.9)$$

and

$$\frac{\langle x_i^{(1)}(t)\, x_i^{(2)}(t)\rangle}{\langle x_j^{(1)}(t)^2\rangle^{\frac{1}{2}}\langle x_k^{(2)}(t)^2\rangle^{\frac{1}{2}}} = \frac{F_2(t,t)}{F(t,t)} = \frac{\mathscr{C}_1 - \frac{1}{3}\mathscr{C}_2}{\mathscr{C}_0} \quad (t_0 \ll t \ll t_L). \tag{4.10}$$

These are thus both constant within the intermediate subrange, showing explicitly that the two-particle effects are of the same order as one-particle effects.

4.3. *Further constraints*

We show in Appendix B, that two-point Lagrangian correlations for any quantity (e.g. acceleration, velocity etc.) satisfy the inequalities

$$0 \leqslant \rho_2(t,t) \leqslant 1 \tag{4.11}$$

and

$$\rho_2^2(t_1,t_2) \leqslant \rho_2(t_1,t_1)\,\rho_2(t_2,t_2) \leqslant 1, \tag{4.12}$$

where ρ_2 is a two-particle correlation coefficient. For acceleration correlations with ρ_2 given by $R_2(t_1,t_2)/(a_0\,\bar{\epsilon}/t_\eta)$ where $a_0\,\bar{\epsilon}/t_\eta$ is the acceleration variance (see figure 1), these inequalities in the intermediate subrange require that

$$0 \leqslant \mathscr{R}_2(1) \leqslant a_0\,t/t_\eta \tag{4.13}$$

and

$$|\mathscr{R}_2(\xi)| \leqslant \xi^{\frac{1}{2}}\mathscr{R}_2(1). \tag{4.14}$$

Thus $\mathscr{R}_2(1)$ is not negative and clearly $\mathscr{R}_2(\xi)$ is tangent to $\xi^{\frac{1}{2}}\mathscr{R}_2(1)$ at $\xi = 1$, i.e.

$$\left.\frac{\mathrm{d}\mathscr{R}_2(\xi)}{\mathrm{d}\xi}\right|_{\xi=1} = \tfrac{1}{2}\mathscr{R}_2(1)$$

which can also be shown to be a consequence of symmetry.

For velocity difference correlations, $D_2(t_1,t_2)/(D(t_1)\,D(t_2))^{\frac{1}{2}}$, we have

$$0 \leqslant \tfrac{2}{3}\mathscr{C}_1/\mathscr{C}_0 \leqslant 1 \tag{4.15}$$

and

$$|\mathscr{D}_2(\xi)| \leqslant \xi^{-\frac{1}{2}}\mathscr{D}_2(1), \tag{4.16}$$

with the tangency condition

$$\left.\frac{\mathrm{d}\mathscr{D}_2(\xi)}{\mathrm{d}\xi}\right|_{\xi=1} = -\tfrac{1}{2}\mathscr{D}_2(1),$$

again a consequence of symmetry.

Finally, for displacement correlations, $F_2(t_1,t_2)/(F(t_1)\,F(t_2))^{\frac{1}{2}}$,

$$0 \leqslant (\mathscr{C}_1 - \tfrac{1}{3}\mathscr{C}_2)/\mathscr{C}_0 \leqslant 1, \tag{4.17}$$

$$|\mathscr{F}_2(\xi)| \leqslant \xi^{-\frac{3}{2}}\mathscr{F}_2(1), \tag{4.18}$$

and the tangency condition

$$\left.\frac{\mathrm{d}\mathscr{F}_2(\xi)}{\mathrm{d}\xi}\right|_{\xi=1} = -\tfrac{3}{2}\mathscr{F}_2(1).$$

Note that these constraints strengthen the usual Schwarz inequalities, which are included in the more restrictive inequalities (4.13), (4.15) and (4.17). However, (4.14), (4.16) and (4.18) are apparently entirely new or at least little known.

It is relatively simple to find a functional form for $\mathscr{R}_2(\xi)$ which satisfies all of the constraints so far considered: say

$$\mathscr{R}_2(\xi) = \alpha\frac{\xi^{\mu}}{1+\xi^{2\mu-1}}, \tag{4.19}$$

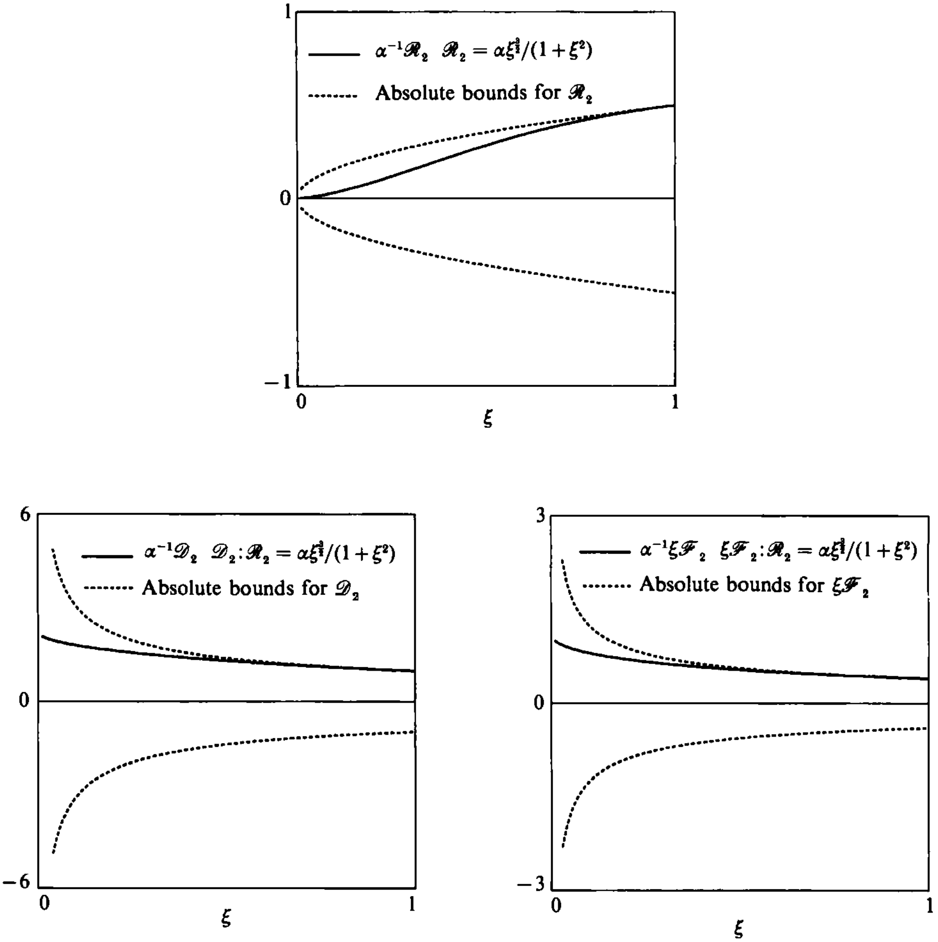

FIGURE 3. Inertial-range forms corresponding to (4.19) with $\mu = \frac{3}{2}$; the acceleration covariance is appropriate for large t_1/t_0 and t_2/t_0 in figure 2. The absolute bounds for these functions are as determined in §4: (4.14), (4.16) and (4.18).

where α and μ are some positive constants with $\mu > 1$. In figure 3 calculated forms for $\mathscr{R}_2$, $\mathscr{D}_2$ and $\mathscr{F}_2$, and the respective bounds, are shown for $\mu = \frac{3}{2}$. Note that the important dispersion characteristics, $\mathscr{C}_1$ and $\mathscr{C}_2$, are given by

$$\mathscr{C}_1 = \frac{\alpha}{2\mu-1}\beta\left(\frac{\mu}{2\mu-1}\right), \quad \mathscr{C}_2 = \frac{\alpha}{2\mu-1}\beta\left(\frac{\mu+1}{2\mu-1}\right), \tag{4.20}$$

where $\beta(q)$ is discussed in Gradshteyn & Ryzhik (1980, p. 947). For the particular case of $\mu = \frac{3}{2}$ it follows that

$$\mathscr{C}_1/\alpha = 0.4875..., \quad \mathscr{C}_2/\alpha = 0.26605... \tag{4.21}$$

and these numbers will be useful in a subsequent section.

The point of example (4.19) is that all the constraints so far considered do not preclude its existence. In particular, there is no kinematic requirement that the two-particle acceleration covariance be trivial in the inertial range as there was in the one-particle case. Of course, it is unlikely that the actual acceleration-covariance intermediate subrange will correspond to such a simple form as (4.19) though it may be that for some values of μ and α it constitutes a reasonable approximation of that behaviour. Unfortunately, there is no constraint that positively establishes that the acceleration-covariance intermediate subrange must be non-trivial (i.e. $\alpha \neq 0$) but it seems only reasonable to suppose that it does exist because of the absence of constraints requiring otherwise.

5. Implications for relative dispersion

We recover results for relative statistics by combining one- and two-particle forms as in (2.15). Thus for small times in the inertial range where two-particle effects are negligible (cf. (4.2)), we have

$$D^{(r)} \approx 3\mathscr{C}_0 \bar{\epsilon}\{t_1+t_2-|t_2-t_1|\} \quad (t_1, t_2 \ll t_0) \tag{5.1}$$

which, remembering that (5.1) refers to the dot-product of velocity differences, is just twice the one-particle result (3.19). However, as we have shown, for intermediate times the two-particle terms are significant and we have

$$D^{(r)} \approx 3\mathscr{C}_0 \bar{\epsilon}\{t_1+t_2-|t_2-t_1|\} - 2\bar{\epsilon}t_1 \mathscr{D}_2\left(\frac{t_1}{t_2}\right) \quad (t_0 \ll t_1, t_2 \ll t_L) \tag{5.2}$$

where both terms in (5.2) are formally of the same order of magnitude although they have their origins in accelerations on vastly different scales. The particular result for mean-square relative velocity differences follows from (5.1) and (5.2) by putting $t_1 = t_2$,

$$\langle \mathscr{u}_i^{(r)}(t)^2\rangle \approx \langle u_i^{(r)}(t)^2\rangle - \langle u^{(r)}(0)^2\rangle \approx \begin{cases} 6\mathscr{C}_0\bar{\epsilon}t & (t_\eta \ll t \ll t_0) \\ (6\mathscr{C}_0 - 4\mathscr{C}_1)\bar{\epsilon}t & (t_0 \ll t \ll t_L), \end{cases} \tag{5.3}$$

where we have also expressed the relative velocity difference variance in terms of the more familiar relative velocity variance. Thus, in the intermediate subrange the relative-velocity variance is influenced significantly by two-particle effects and is not trivially related to the one-particle variance (by a factor of two) as concluded by Novikov (1963) and Monin & Yaglom (1975). Of course, the exponent in the power law is not altered, since simple dimensional analysis prevents such a discrepancy, but the coefficients differ.

Similar conclusions apply for the relative displacements. There will generally be two subranges of the inertial range which correspond to the effects of one- and two-particle acceleration correlations. In particular, both of these subranges are characterized by cubic growth with t (along the diagonal) with respective coefficients

$$\mathscr{F}^{(r)}(1) = \begin{cases} 2\mathscr{C}_0 & \text{when} \quad t_\eta \ll t \ll t_0 \\ 2\mathscr{C}_0 - 2\mathscr{C}_1 + \frac{2}{3}\mathscr{C}_2 & \text{when} \quad t_0 \ll t \ll t_L. \end{cases} \tag{5.4}$$

The corresponding dimensional forms for the relative dispersion are

$$\langle x_i^{(r)}(t)^2\rangle \approx \langle \Delta_i(t)^2\rangle - \langle u_i^{(r)}(0)^2\rangle t^2 - \Delta_0^2 \approx \begin{cases} 2\mathscr{C}_0\bar{\epsilon}t^3 & (t_\eta \ll t \ll t_0) \\ (2\mathscr{C}_0 - 2\mathscr{C}_1 + \frac{2}{3}\mathscr{C}_2)\bar{\epsilon}t^3 & (t_0 \ll t \ll t_L), \end{cases} \tag{5.5}$$

where we have also expressed these results in terms of the particle separation, Δ. Again, the coefficients differ but the power-law exponent does not.

6. Calculations with a Markov model

Dimensional arguments advanced so far cannot prescribe the form of the functions $\mathscr{R}_2(\xi)$, $\mathscr{D}_2(\xi)$ and $\mathscr{F}_2(\xi)$, nor can they show that the constants $\mathscr{C}_1$ and $\mathscr{C}_2$ are non-zero. In this section we present evidence to show that $\mathscr{C}_1$ and $\mathscr{C}_2$ *are* non-zero and that therefore the function $\mathscr{R}_2(\xi)$ is non-trivial.

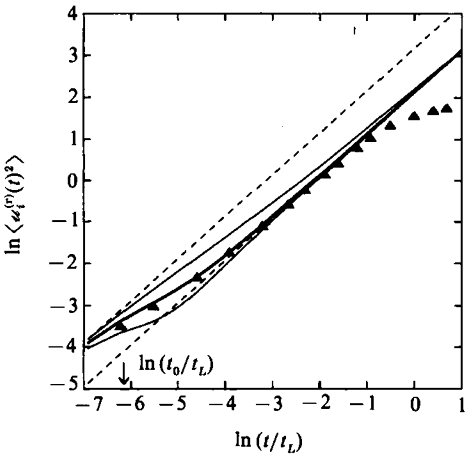

FIGURE 4. Relative-velocity-increment variance as a function of time. Stochastic model results (▲) are compared with various parameterizations of the two-particle two-time acceleration covariance: ——, (6.2) with $\mu = \frac{3}{2}$; ——, (6.2) with $\mu = \frac{1}{2}$ (upper line) and $\mu = \frac{5}{2}$ (lower line); ---, inertial-range form (5.3) for $t \ll t_0$ (upper line) and $t \gg t_0$ (lower line).

An early Markov model of Novikov (1963) for *relative* dispersion is a direct analogue of a one-particle model but with the random initial velocity chosen so that the initial (Eulerian) relative-velocity statistics are correct. That is, according to Novikov's model,

$$D_2(t,t) = \langle u_i^{(1)}(0)\, u_i^{(2)}(0)\rangle \left[\exp\left(-\frac{t}{t_L}\right) - 1\right]^2$$

which to leading order within the inertial range is

$$D_2(t,t) \approx \bar{\epsilon} t^2/t_L \quad (t_0 \ll t \ll t_L). \tag{6.1}$$

Comparison with the correct small-time result (4.2) and the new intermediate-time results (4.8*b*), shows that (6.1) is too small by factors of order t_0/t_L and t/t_L respectively, where both t_0 and $t \ll t_L$. Similarly, this model has a two-particle acceleration covariance, which being of $O(\bar{\epsilon}/t_L)$ is also too small implying that $\mathscr{R}_2 \approx 0$ even for very small separations. Therefore this model involves an unnecessarily restrictive assumption.

A more recent Markov model (Thomson 1990), which is a more sophisticated extension of Novikov's model, can more generally represent the acceleration covariance and exhibits non-trivial two-particle dispersion. This can be seen in figures 4 and 5 for the mean-square velocity difference and displacements respectively where results of a numerical simulation of 2×10^4 realisations with $\mathscr{C}_0 = 4$ and an initial separation of $\Delta_0 = 10^{-4}$ are given. The model also requires as input the Eulerian two-point velocity covariance function, which in the inertial range is of the form

$$\langle u_i(\boldsymbol{x}) u_i(\boldsymbol{x}+\Delta_0)\rangle \approx 3\sigma^2 - \tfrac{11}{3} C(\bar{\epsilon}\Delta_0)^{\frac{2}{3}} \quad (\Delta_0 \ll L),$$

where $C = 1$ (Townsend 1976, pp. 96–99). The upper straight dashed line in figures 4 and 5 shows the small-time behaviour, (5.3) and (5.4) respectively, for $t \ll t_0$ from which the numerical solution departs when $t \approx t_0$. However, the numerical results approach a second straight line, the lower dashed line, in what we have termed the

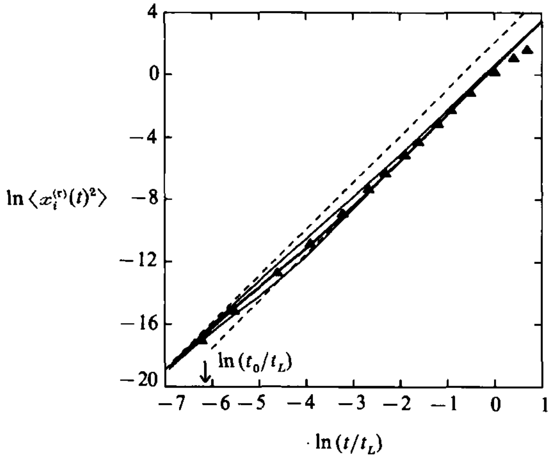

FIGURE 5. As for figure 4, except that here relative-displacement statistics are shown and the dashed line is inertial-range form (5.4).

intermediate range, which corresponds to the numerical values $\mathscr{C}_1 = 3.9$ and $\mathscr{C}_2 = 2.1$ in (5.3) and (5.4). Eventually, for $t \approx t_L$ the simulation deviates from this second asymptotic regime.

Thomson's model implicitly accounts for both the small-time and intermediate-time behaviour and for the transition between these two ranges. Here we propose a suitable interpolation for the two-particle acceleration covariance which encompasses (4.19) and the small-time behaviour $\mathscr{R}_2 \approx \kappa\bar{\epsilon}/t_0$, say

$$R_2(t_1, t_2) = \alpha\bar{\epsilon}\,\frac{(t_1^2+t_0^2)^{\frac{1}{2}(\mu-1)}\,(t_2^2+t_0^2)^{\frac{1}{2}(\mu-1)}}{(t_1^2+t_0^2)^{\mu-\frac{1}{2}}+(t_2^2+t_0^2)^{\mu-\frac{1}{2}}} \tag{6.2}$$

where (6.2) is chosen to be symmetric about the t_1 and t_2 axes and we have chosen $\kappa \approx \alpha$. (Figure 2 was generated from (6.2) with $\mu = \frac{3}{2}$.) We estimate α by a fit to the stochastic-model simulation of the intermediate-range mean-square velocity; i.e. (4.21) fits the numerical result $\mathscr{C}_1 \approx 3.9$ with $\alpha \approx 8$. The mean-square velocity corresponding to (6.2) with $\mu = \frac{3}{2}$ (obtained by integrating (2.16) numerically) is shown in figure 4 as the heavy solid line; it corresponds to the Markov results very well. Two other values of the exponent μ (with a similar fitting procedure in the intermediate range) were also tried and are shown in figure 4, but do not represent the transition between the small- and intermediate-time results as well as $\mu = \frac{3}{2}$. Note that the inertial-range form (6.2) is not truncated at large times, so its intermediate range continues indefinitely whereas the stochastic simulations show that for t/t_L large the velocity-difference covariance tends to a constant.

A more searching test of the suitability of (6.2) is to now consider figure 5 and the displacement statistics since there are no more adjustable parameters in (6.2). Again we find excellent agreement with the Markov results for $\mu = \frac{3}{2}$, and a less satisfactory fit for $\mu = \frac{1}{2}$ or $\frac{5}{2}$, from which we conclude that (6.2) captures the essential ingredients of the acceleration covariance in the inertial range which are implicit in Thomson's stochastic equations.

We believe Thomson's model is a more faithful representation of the physics (at least in comparison with Novikov's model). For example, it meets objections raised in Novikov (1989) to such particle-trajectory models. However, this model does not explicitly ensure that the acceleration covariance, which is apparently not too dissimilar to (6.2), is actually correct. In other words (6.2) may in fact be far from the acceleration covariance corresponding to the Navier–Stokes equations. It is clear that a future course is to design a model which properly incorporates the acceleration covariance.

There are several predictions in the literature which have some indirect bearing on the issue of the importance of the two-particle acceleration covariance. For example, Larchevêque & Lesieur (1981) used an EDQNM closure of the equations of motion to calculate the coefficient in the relative dispersion t^3-law. They obtained a value $\alpha_{\mathrm{Ri}} \approx 3.5$ for Richardson's constant, which in our notation corresponds to

$$\alpha_{\mathrm{Ri}} = 2\mathscr{C}_0 - 2\mathscr{C}_1 + \tfrac{2}{3}\mathscr{C}_2 \approx 3.5.$$

Similarly an earlier Lagrangian History Direct Interaction analysis by Kraichnan (1966*b*) yields a value $\alpha_{\mathrm{Ri}} = 2.42$. There are difficulties (and consequent inconsistencies) in any attempt to derive one-particle statistics from either of these models since the limit $\Delta_0 \rightarrow 0$ merely causes the timescale t_0 to vanish thus effectively extending the intermediate subrange to arbitrarily small times. However, it is possible to derive from Kraichnan's one-particle *displacement* statistics a value of $\mathscr{C}_0 = 4.67$, and supposing a $\mathscr{C}_0$ value similarly greater than 2 for the EDQNM model, then both results imply that $\mathscr{C}_1$ and $\mathscr{C}_2$ are non-zero and that Richardson's constant is not trivially related to $\mathscr{C}_0$.

A more recent attempt by Hunt *et al.* (1990), to model relative dispersion in turbulence, where a kinematic flow field is specified, gives the result $\alpha_{\mathrm{Ri}} \approx 0.3$, which is so low as to suggest that in this case the additional two-particle constants $\mathscr{C}_1$ and $\mathscr{C}_2$ contrive to almost cancel with $2\mathscr{C}_0$ in (5.5). For these kinematic simulations, $\mathscr{C}_0 = 2.2$.

7. Conclusion

In this paper we have examined the structure of one- and two-particle Lagrangian turbulence statistics, focusing particularly on the inertial subrange in the limit of large Reynolds number. Our findings differ from presently accepted results in a number of ways.

For one-particle statistics, we showed that the inertial-subrange form of the Lagrangian acceleration covariance derived from dimensional arguments, $\mathscr{C}_0^{(a)}\bar{\epsilon}/\tau$, is kinematically inconsistent with the corresponding velocity statistics unless $\mathscr{C}_0^{(a)} = 0$.

One-particle velocity and displacement statistics (in an inertial frame moving with the initial velocity of the particle in each realization) are non-trivial in the inertial subrange, and the traditional results obtained by dimensional analysis are confirmed here. However, through our analysis of asymptotic expansions we have shown that the universal constant $\mathscr{C}_0$ which characterizes these latter statistics in the inertial subrange is entirely prescribed by the leading-order term (in Reynolds number) of the inner expansion of the acceleration covariance, i.e. by the dissipation-range structure of the acceleration covariance.

The direct matching of large-scale acceleration structure to that in the dissipation range is particularly apparent in the limit of infinite Reynolds number. In that limit, we have shown that the outer structure (involving τ/t_L) is applicable to all non-zero lags and the inner or dissipation-range structure is compressed into a δ-function at the $\tau = 0$ origin. This is precisely the form for the acceleration covariance which is obtained from Langevin models of the motion of marked particles and which are currently of great interest.

In the two-particle case the acceleration covariance is non-stationary and therefore a function of the two times t_1 and t_2 rather than the lag $|t_1 - t_2|$. In addition it is a function of the initial separation (through the timescale t_0) and the turbulence timescales t_η and t_L. For $\eta \ll \Delta_0 \ll L$, dissipation-scale effects can be ignored and we have focused on a part of the inertial subrange for which $t_0 \ll t_1, t_2 \ll t_L$. There dimensional arguments require the acceleration covariance to be of the form $\bar{\epsilon} t_1^{-1} \mathscr{R}_2(t_1/t_2)$. In contrast to the one-particle case, there is no kinematic inconsistency between this form and the corresponding results for velocity and displacement statistics, i.e. there is no constraint causing $\mathscr{R}_2(\xi)$ to vanish.

An important consequence of this non-trivial form for $\mathscr{R}_2(\xi)$ is that two-particle corrections to the inertial-subrange structure of the relative velocity and displacement covariances cannot be ignored. These corrections do not affect the power-law dependence of these quantities within the inertial subrange but do alter the magnitude of the constant of proportionality. Thus the relative-velocity variance is of the form $(6\mathscr{C}_0 - 4\mathscr{C}_1)\bar{\epsilon}\tau$, where

$$\mathscr{C}_1 = \int_0^1 \xi^{-1} \mathscr{R}_2(\xi)\,\mathrm{d}\xi,$$

and the relative dispersion is of the form $(2\mathscr{C}_0 - 2\mathscr{C}_1 + \frac{2}{3}\mathscr{C}_2)\bar{\epsilon}\tau^3$, where

$$\mathscr{C}_2 = \int_0^1 \mathscr{R}_2(\xi)\,\mathrm{d}\xi.$$

These results differ by the two-particle corrections from those previously accepted and which result from the assumption that the two-particle acceleration covariance can be ignored. They thus destroy the simple correspondence between relative- and one-particle statistics traditionally derived from that assumption. There is some evidence from other analyses of relative dispersion to show that these constants do not vanish.

In general $\mathscr{R}_2(\xi)$ remains undetermined, although some further inequality constraints, as outlined in Appendix B, can be placed on it. These do not appear to be well known and constrain the form of $\mathscr{R}_2(\xi)$ so that, in particular, the constants $\mathscr{C}_1$ and $\mathscr{C}_1 - \frac{1}{3}\mathscr{C}_2$ are non-negative and consequently the relative dispersion proceeds less rapidly than if independence of the particles' motion is assumed.

Our analysis has clarified the role of two-particle effects in the process of relative dispersion and provides more rigorous support for Thomson's (1990) empirical results with a Lagrangian stochastic model of two-particle motion. We are hopeful that our results will aid the design of such models so that they properly model the acceleration covariance.

Appendix A

The dynamical equations of motion for a fluid are the Navier–Stokes equations,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{A 1}$$

and the equation of continuity,

$$\frac{\partial u_i}{\partial x_i} = 0$$

The left-hand side of (A 1) is the acceleration of a fluid particle at position $\boldsymbol{x}$ and with velocity $\boldsymbol{u}$ due to a pressure gradient and a viscous force. Generally the flow is generated by some boundary conditions or forcing. For instance, some forcing that does work on the fluid at the rate $\bar{\epsilon}$ per unit mass. However, for this discussion the velocity field is supposed to exist and (A 1) will be used to consider the consequent accelerations.

Consider the following exact relationship:

$$\int_0^\infty \langle a_i^{(1)}(t_1)\, a_j^{(2)}(t_2)\rangle \, \mathrm{d}t_1 = -\langle u_i^{(1)}(0)\, a_j^{(2)}(t_2)\rangle, \tag{A 2}$$

where the notation is from the main body of the text. In particular, set $t_2 = 0$ so that the right-hand side is simply a function of the initial (fixed) separation: Δ_0.

The correlation between the acceleration of a particle at position $\boldsymbol{x}'$ and the velocity of a particle at $\boldsymbol{x}$, where $|\boldsymbol{x}'-\boldsymbol{x}| = |\Delta| \gg \nu^{\frac{3}{4}}\bar{\epsilon}^{-\frac{1}{4}}$ ($=\eta$, the viscous lengthscale), is approximately determined by the two-point pressure-gradient/velocity correlation because the high shears are not well correlated over large distances. Therefore

$$\langle u_i^{(1)}(0)\, a_j^{(2)}(0)\rangle \approx -\rho^{-1}\left\langle u_i^{(1)}(\boldsymbol{x}_0^{(1)},0)\frac{\partial p(\boldsymbol{x}_0^{(2)},0)}{\partial x_{0j}^{(2)}}\right\rangle = Q_{ij}(\Delta_0), \tag{A 3}$$

which defines the tensor $\boldsymbol{Q}$. $p(\boldsymbol{x}_0^{(2)},0)$ is an Eulerian function of $\boldsymbol{x}_0^{(2)}$ at time $t=0$ which represents the initial distribution of the pressure field. Provided that $|\Delta_0| \gg \eta$ the corrections in (A 3) due to the viscous stresses will be negligible.

$\boldsymbol{Q}$ has special properties: the first due to mass continuity

$$\frac{\partial Q_{ij}(\Delta_0)}{\partial x_{0i}^{(1)}} = -\frac{\partial Q_{ij}(\Delta_0)}{\partial \Delta_{0i}} = 0 \;\forall\, j = 1,2,3;$$

and the second due to the curl-free dynamic acceleration, thus

$$\epsilon_{ljk}\frac{\partial Q_{ij}(\Delta_0)}{\partial x_{0k}^{(2)}} = \epsilon_{ljk}\frac{\partial Q_{ij}(\Delta_0)}{\partial \Delta_{0k}} = 0 \;\forall\, i,l = 1,2,3$$

where $\epsilon_{ijk} = 1$ if $\{ijk\}$ is a cyclic permutation of $\{123\}$, but $\epsilon_{ijk} = 0$ if any two indices are the same and is otherwise equal to -1.

As a result of these two properties, and the further properties that, firstly, $\boldsymbol{Q}$ is a bounded function of Δ_0 and, secondly, of isotropy, the tensor $\boldsymbol{Q}$ is at most a constant function of Δ_0. However, that constant is necessarily zero because the correlation for arbitrarily large separations must be asymptotically small. Therefore the main result is that

$$\int_0^\infty \langle a_i^{(1)}(t_1;\boldsymbol{x}_0^{(1)})\, a_j^{(2)}(0;\boldsymbol{x}_0^{(2)})\rangle \, \mathrm{d}t_1 \approx 0 \;\forall\, i,j = 1,2,3, \tag{A 4}$$

with corrections being dependent on the viscous terms and therefore Reynolds-number dependent,

Now let $\langle a_i^{(1)}(t_1)\, a_i^{(2)}(0)\rangle = R_0(t_1)$. Thus (A 4) is equivalent to

$$\int_0^\infty R_0(t)\, \mathrm{d}t = 0 \tag{A 5}$$

which is a direct analogue of the one-particle kinematic relationship (3.5). Therefore (with t_0 now playing the role of the inner timescale) it is not possible that $R_0(t)$ has the naive dimensional behaviour, $R_0(t) \approx \bar{\epsilon}/t$, for $t \gg t_0$ in the inertial range. Thus $R_0(t)$ vanishes faster than inversely with t in the inertial range. Consequently, $\mathscr{R}_2(\xi)$ as defined in (4.3) behaves like $\xi^{1+\lambda}$ for small ξ and some λ (>0).

Appendix B

Here we consider constraints on various two-particle covariances using an argument essentially due to Thomson (1990) (appendix B) but with origins based on Batchelor (1952).

We begin with one-time two-particle covariances (such as $R_2(t,t)$, $D_2(t,t)$, $F_2(t,t)$ etc.). Consider a collection of N marked fluid particles at time t (>0) which at time $t=0$ were distributed within a cloud of radius Δ_0, where $\Delta_0 \ll L$. Now consider an ensemble of such situations where in each the particles are dispersed by homogeneous, isotropic and stationary turbulence (with energy-containing lengthscale L). Now the statistics of different pairs are distinguishable only through their initial separations. Furthermore, for times such that $t_0 = \bar{\epsilon}^{-\frac{1}{3}}\Delta_0^{\frac{2}{3}} \ll t$, the statistics of all pairs are indistinguishable.

Consider now the $N \times N$ pair correlation matrix, $\boldsymbol{C}(t)$, for an arbitrary quantity (say the displacement, velocity or acceleration of each particle). Since all pairs are equivalent, $\boldsymbol{C}$ has a determinant of the form

$$|\boldsymbol{C}(t)| = \begin{vmatrix} 1 & \rho & \rho & & \cdots & \rho \\ \rho & 1 & \rho & & \cdots & \rho \\ \rho & \rho & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & 1 & \rho \\ \rho & \rho & \cdots & & \rho & 1 \end{vmatrix}, \tag{B 1}$$

where, for displacements say,

$$\rho(t) = \frac{\langle x_i^{(\mathrm{i})}(t)\, x_i^{(\mathrm{j})}(t)\rangle}{\langle x_j^{(\mathrm{i})}(t)\, x_j^{(\mathrm{i})}(t)\rangle^{\frac{1}{2}} \langle x_k^{(\mathrm{j})}(t)\, x_k^{(\mathrm{j})}(t)\rangle^{\frac{1}{2}}}$$

and is independent of the particle labelling (i) and (j). Since a correlation matrix must have a non-negative determinant.

$$|\boldsymbol{C}(t)| = (1-\rho)^{N-1}(1+(N-1)\rho) \geqslant 0. \tag{B 2}$$

We may take N arbitrarily large, so that we have finally

$$0 \leqslant \rho \leqslant 1,$$

which is the main one-time result.

We now treat two-time two-particle covariances such as $R_2(t_1,t_2)$, $D_2(t_1,t_2)$, $F_2(t_1, t_2)$, by considering the analogous problem for $2N$ particles, N at time t_1 and N at time t_2 but all, as above, emanating from a cloud initially of radius Δ_0. Again pair statistics are indistinguishable when both t_1 and t_2 are much larger than t_0. The correlation matrix, $\boldsymbol{C}_2(t_1,t_2)$, now contains three sorts of off-diagonal elements; one-time correlations $\rho' = \rho(t_1)$ and $\rho'' = \rho(t_2)$ defined as above, and a two-time correlation, which, for example, for displacements, is given by

$$\rho_2(t_1,t_2) = \frac{\langle x_i^{(\mathrm{i})}(t_1)\, x_i^{(\mathrm{j})}(t_2)\rangle}{\langle x_j^{(\mathrm{i})}(t_1)\, x_j^{(\mathrm{i})}(t_1)\rangle^{\frac{1}{2}} \langle x_k^{(\mathrm{j})}(t_2)\, x_k^{(\mathrm{j})}(t_2)\rangle^{\frac{1}{2}}}.$$

$\boldsymbol{C}_2$ again has a non-negative determinant,

$$
|\boldsymbol{C}_2(t_1,t)| = \left|\begin{array}{cccccc|cccccc}
1 & \rho' & \rho' & \cdots & & \rho' & \rho_2 & \rho_2 & \rho_2 & \cdots & & \rho_2 \\
\rho' & 1 & \rho' & \cdots & & \rho' & \rho_2 & \rho_2 & \rho_2 & \cdots & & \rho_2 \\
\rho' & \rho' & 1 & & & \vdots & \rho_2 & \rho_2 & \rho_2 & \ddots & & \vdots \\
\vdots & & & & 1 & \rho' & \vdots & & & & \rho_2 & \rho_2 \\
\rho' & \rho' & \cdots & & \rho' & 1 & \rho_2 & \rho_2 & \cdots & & \rho_2 & \rho_2 \\
\hline
\rho_2 & \rho_2 & \rho_2 & \cdots & & \rho_2 & 1 & \rho'' & \rho'' & \cdots & & \rho'' \\
\rho_2 & \rho_2 & \rho_2 & \cdots & & \rho_2 & \rho'' & 1 & \rho'' & \cdots & & \rho'' \\
\rho_2 & \rho_2 & \rho_2 & \ddots & & \vdots & \rho'' & \rho'' & 1 & \ddots & & \vdots \\
\vdots & & & \ddots & \rho_2 & \rho_2 & \vdots & & & \ddots & 1 & \rho'' \\
\rho_2 & \rho_2 & \cdots & & \rho_2 & \rho_2 & \rho'' & \rho'' & \cdots & & \rho'' & 1
\end{array}\right| \geqslant 0
$$

Expanding the determinant gives

$$|\boldsymbol{C}_2| = (1-\rho')^{N-1}(1-\rho'')^{N-1}(1-(2N-1)\rho'\rho''+(N-1)(\rho'+\rho'')+N^2(\rho'\rho''-\rho_2^2)).$$

Hence non-negativity (for arbitrarily large N) requires

$$\rho(t_1)\,\rho(t_2) \geqslant \rho_2^2(t_1,t_2)\,\forall\, t_1, t_2 \tag{B 3}$$

since we already have $0 \leqslant \rho(t_1),\ \rho(t_2) \leqslant 1$.

Equation (B 3) is the main two-time result.

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